Simplification of metric spaces

Metric graph approximations

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- What can we say about the approximation if we put an upper bound the first Betti number of the approximating graphs ?
- Given a compact geodesic space X, we define the sequence $(\delta_n^X)_{n\geq 0}$ as follows :

$$\delta_n^X := \inf\{d_{\mathrm{GH}}(X,G) : G \text{ a finite metric graph}, \beta_1(G) \leq n\}.$$

Approximation by the Reeb graph

- Given a function $f : X \to \mathbb{R}$, the Reeb graph X_f is the quotient space $X/\sim$ where $x \sim y$ if there is a continuous path between x and y on which f is constant. This is a graph under certain conditions and it can be given a length structure pulled back by f. If $f = d(p, \cdot)$ for some p in X, then we denote X_f by X_p. It is known that $\beta_1(X_p) \leq \beta_1(X)$.

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Let X be a compact geodesic space such that $\beta = \beta_1(X)$ is finite and p be a point in X. Then,

i) *For $n \geq \beta$,*

$$\frac{d_{\mathrm{GH}}(X, X_p)}{16n + 13} \leq \delta_n^X \leq d_{\mathrm{GH}}(X, X_p).$$

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ii) *Let $a_1^X \geq a_2^X \geq \ldots$ be the lengths of the intervals in the first persistent barcode of the open Vietoris-Rips filtration of X. For $n < \beta$,*

$$\frac{d_{\mathrm{GH}}(X, X_p)}{16\,\beta + 13} \leq \delta_n^X \leq d_{\mathrm{GH}}(X, X_p) + (6\,\beta + 6)\,a_{n+1}^X.$$